

WAVE EQUATION FOR 2D LANDAU HAMILTONIAN

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ABSTRACT. This note is devoted to the study of the well-posedness of the Cauchy problem for the Landau Hamiltonian wave equation in the plane, with nonzero constant magnetic field. We show the well-posedness in suitably defined Sobolev spaces taking into account the spectral properties of the operator.

1. INTRODUCTION

The Landau Hamiltonian describing the motion of the quantum particle in two dimensions in the constant magnetic field is one of the classical models in quantum physics. It was introduced and studied in late 1920s, see Landau and Lifshitz [LL77]. It turned out that the system possesses rather unusual spectral properties: the spectrum consists of discrete eigenvalues of infinite multiplicities lying at the points of an arithmetic progression. These eigenvalues are traditionally called Landau levels and the corresponding eigenspaces are called Landau subspaces.

The dynamics of a particle with charge e and mass m_* on the Euclidean xy -plane interacting with a perpendicular constant homogeneous magnetic field is given by the Hamiltonian operator

$$(1.1) \quad \mathcal{H}_0 := \frac{1}{2m_*} \left(ih\nabla - \frac{e}{c} \mathbb{A} \right)^2,$$

where h is the Planck's constant, c is the speed of light, and i is the imaginary unit. Denoting by $2B > 0$ the strength of the magnetic field and selecting the symmetric gauge

$$\mathbb{A} = -\frac{\mathbf{r}}{2} \times 2B = (-By, Bx),$$

where $\mathbf{r} = (x, y) \in \mathbb{R}^2$, as well as setting $m_* = e = c = h = 1$, we get the Landau Hamiltonian

$$(1.2) \quad \mathcal{H} := \frac{1}{2} \left(\left(i\frac{\partial}{\partial x} - By \right)^2 + \left(i\frac{\partial}{\partial y} + Bx \right)^2 \right),$$

acting on the Hilbert space $L^2(\mathbb{R}^2)$. It is well known ([F28, L30]) that the operator \mathcal{H} has the spectrum consisting of infinite number of eigenvalues with infinite multiplicity,

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of the form

$$(1.3) \quad \lambda_n = (2n + 1)B, \quad n = 0, 1, 2, \dots,$$

and these eigenvalues are called the Euclidean Landau levels. Let us denote by $\mathcal{A}_n(\mathbb{R}^2)$ the eigenspace of \mathcal{H} corresponding to the eigenvalue λ_n in (1.3):

$$(1.4) \quad \mathcal{A}_n(\mathbb{R}^2) = \{\varphi \in L^2(\mathbb{R}^2), \mathcal{H}\varphi = \lambda_n\varphi\}.$$

The following functions form an orthogonal basis for $\mathcal{A}_n(\mathbb{R}^2)$ (see [ABGM15] for more details on this and the above, and also [HH13]):

$$(1.5) \quad \begin{cases} e_{k,n}^1(x, y) = \sqrt{\frac{n!}{(n-k)!}} B^{\frac{k+1}{2}} \exp\left(-\frac{B(x^2 + y^2)}{2}\right) (x + iy)^k L_n^{(k)}(B(x^2 + y^2)), & 0 \leq k, \\ e_{j,n}^2(x, y) = \sqrt{\frac{j!}{(j+n)!}} B^{\frac{n+1}{2}} \exp\left(-\frac{B(x^2 + y^2)}{2}\right) (x - iy)^n L_j^{(n)}(B(x^2 + y^2)), & 0 \leq j, \end{cases}$$

where $L_n^{(\alpha)}$ is the Laguerre polynomial defined as

$$L_n^{(\alpha)}(t) = \sum_{k=0}^n (-1)^k C_{n+\alpha}^{m-k} \frac{t^k}{k!}, \quad \alpha > -1.$$

To simplify further notation we denote

$$(1.6) \quad e_\xi^k := e_{i,n}^k \quad \text{for } \xi = (i, n), \quad i, n = 0, 1, 2, \dots; \quad k = 1, 2.$$

The basis (1.5) has relations to many other subjects. For example, see [I16] for relations to the complex Hermite polynomials, [ABG12, BG14, CGG10] for relations to quantization problems, [A10] for their appearance in the time-frequency analysis, [G08] in partial differential equations, and [HH13] for planar point processes. We refer to [ABGM15] for further details, as well as to a relation to Feynman and Schwinger's work on the displacement operator, see also Perelomov [P86, p. 35].

We can also refer to papers [K16, N96], where the authors investigated properties of eigenfunctions of perturbed Hamiltonians, with descriptions of the asymptotics of the eigenvalues for perturbed Landau Hamiltonians in [S14, KP04, M91, PR07, PRV13, LR14, RT08].

In this paper, for a continuous propagation speed function $a = a(t) \geq 0$, we will be concerned with the Cauchy problem for the Landau Hamiltonian \mathcal{H} :

$$(1.7) \quad \begin{cases} \partial_t^2 u(t, x) - a(t)\mathcal{H}u(t, x) = 0, & (t, x) \in [0, T] \times \mathbb{R}^2, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^2, \\ \partial_t u(0, x) = u_1(x), & x \in \mathbb{R}^2. \end{cases}$$

Although the equation (1.7) is strictly hyperbolic, the coefficients of \mathcal{H} depend on the space variables and are increasing. Therefore, although we know that the Cauchy problem (1.7) is locally well-posed for $a \in C^1$, it is interesting to establish global in space properties of its solutions. This will be done in terms of the spectral properties of the operator \mathcal{H} .

The description of appearing function spaces is carried out in the spirit of [DR16] using the general development of nonharmonic type analysis carried out by the authors in [RT16]. The treatment of the global well-posedness in the appearing function

spaces is an extension of the method developed in [GR15] in the context of compact Lie groups.

2. THE MAIN RESULT

In our results below, concerning the Cauchy problem (1.7), we will aim at carrying out analysis in the strictly hyperbolic case $a(t) \geq a_0 > 0$, $a \in C^1([0, T])$. This is the regular time-non-degenerate case when we obtain the well-posedness in Sobolev spaces $H_{\mathcal{H}}^s$ associated to the operator \mathcal{H} .

Theorem 2.1. *Assume that $a \in C^1([0, T])$ and that $a(t) \geq a_0 > 0$. For any $s \in \mathbb{R}$, if the Cauchy data satisfy $(u_0, u_1) \in H_{\mathcal{H}}^{1+s} \times H_{\mathcal{H}}^s$, then the Cauchy problem (1.7) has a unique solution $u \in C([0, T], H_{\mathcal{H}}^{1+s}) \cap C^1([0, T], H_{\mathcal{H}}^s)$ which satisfies the estimate*

$$(2.1) \quad \|u(t, \cdot)\|_{H_{\mathcal{H}}^{1+s}}^2 + \|\partial_t u(t, \cdot)\|_{H_{\mathcal{H}}^s}^2 \leq C(\|u_0\|_{H_{\mathcal{H}}^{1+s}}^2 + \|u_1\|_{H_{\mathcal{H}}^s}^2).$$

We now define the Sobolev spaces $H_{\mathcal{H}}^s(G)$ associated to \mathcal{H} that appear in the theorem. Thus, for any $s \in \mathbb{R}$, we set

$$H_{\mathcal{H}}^s(G) := \{f \in \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2) : \mathcal{H}^{s/2} f \in L^2(\mathbb{R}^2)\},$$

with the norm $\|f\|_{H_{\mathcal{H}}^s} := \|\mathcal{H}^{s/2} f\|_{L^2}$. The global space of distributions $\mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2)$ is defined in Section 3. Anticipating the material of the next section, using Plancherel's identity (3.5), in our case we can express the Sobolev norm as

$$(2.2) \quad \|f\|_{H_{\mathcal{H}}^s} = \left(\sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^s \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} f(x) \overline{e_{\xi}^j(x)} dx \right|^2 \right)^{1/2}.$$

3. FOURIER ANALYSIS FOR THE LANDAU HAMILTONIAN

In this section we recall the necessary elements of the global Fourier analysis that has been developed in [RT16]. Although the domain \mathbb{R}^2 in this paper is unbounded, the following constructions carry over without any significant changes. The space $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2) := \text{Dom}(\mathcal{H}^{\infty})$ is called the space of test functions for \mathcal{H} . Here we define

$$\text{Dom}(\mathcal{H}^{\infty}) := \bigcap_{k=1}^{\infty} \text{Dom}(\mathcal{H}^k),$$

where $\text{Dom}(\mathcal{H}^k)$ is the domain of the operator \mathcal{H}^k , in turn defined as

$$\text{Dom}(\mathcal{H}^k) := \{f \in L^2(\mathbb{R}^2) : \mathcal{H}^j f \in L^2(\mathbb{R}^2), \ j = 0, 1, 2, \dots, k-1\}.$$

The Fréchet topology of $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$ is given by the family of norms

$$(3.1) \quad \|\varphi\|_{C_{\mathcal{H}}^k} := \max_{j \leq k} \|\mathcal{H}^j \varphi\|_{L^2(\mathbb{R}^2)}, \quad k \in \mathbb{N}_0, \ \varphi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2).$$

The space

$$\mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2) := \mathcal{L}(C_{\mathcal{H}}^{\infty}(\mathbb{R}^2), \mathbb{C})$$

of linear continuous functionals on $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$ is called the space of \mathcal{H} -distributions. We can understand the continuity here in terms of the topology (3.1). For $w \in \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2)$ and $\varphi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$, we shall write

$$w(\varphi) = \langle w, \varphi \rangle.$$

For any $\psi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$, the functional

$$C_{\mathcal{H}}^{\infty}(\mathbb{R}^2) \ni \varphi \mapsto \int_{\mathbb{R}^2} \psi(x) \varphi(x) dx$$

is an \mathcal{H} -distribution, which gives an embedding $\psi \in C_{\mathcal{H}}^{\infty}(\mathbb{R}^2) \hookrightarrow \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2)$.

Taking into account the fact that the eigenfunctions of the Landau Hamiltonian in (1.5) come in pairs, it will be convenient to group them together in the way suggested by the notation (1.6). This leads to the following definitions. Let $\mathcal{S}(\mathbb{N}^2)$ denote the space of rapidly decaying functions $\varphi : \mathbb{N}^2 \rightarrow \mathbb{C}^{2 \times 2}$ of the form

$$\varphi := \begin{pmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{pmatrix}.$$

That is, $\varphi \in \mathcal{S}(\mathbb{N}^2)$ if for any $M < \infty$ there exists a constant $C_{\varphi, M}$ such that

$$|\varphi(\xi)| \leq C_{\varphi, M} \langle \xi \rangle^{-M}$$

holds for all $\xi \in \mathbb{N}^2$, where we denote

$$\langle \xi \rangle := |\sqrt{\lambda_{\xi_2}}| = |\sqrt{(2\xi_2 + 1)B}|.$$

The topology on $\mathcal{S}(\mathbb{N}^2)$ is given by the seminorms p_k , where $k \in \mathbb{N}_0$ and

$$p_k(\varphi) := \sup_{\xi \in \mathbb{N}^2} \langle \xi \rangle^k |\varphi(\xi)|.$$

We now define the \mathcal{H} -Fourier transform on $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$ as the mapping

$$(\mathcal{F}_{\mathcal{H}} f)(\xi) = (f \mapsto \widehat{f}) : C_{\mathcal{H}}^{\infty}(\mathbb{R}^2) \rightarrow \mathcal{S}(\mathbb{N}^2)$$

by the formula

$$(3.2) \quad \widehat{f}(\xi) := (\mathcal{F}_{\mathcal{H}} f)(\xi) = \int_{\mathbb{R}^2} f(x) \overline{e_{\xi}(x)} dx,$$

where

$$e_{\xi}(x) = \begin{pmatrix} e_{\xi}^1(x) & 0 \\ 0 & e_{\xi}^2(x) \end{pmatrix}.$$

The \mathcal{H} -Fourier transform $\mathcal{F}_{\mathcal{H}}$ is a bijective homeomorphism from $C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$ to $\mathcal{S}(\mathbb{N}^2)$. Its inverse

$$\mathcal{F}_{\mathcal{H}}^{-1} : \mathcal{S}(\mathbb{N}^2) \rightarrow C_{\mathcal{H}}^{\infty}(\mathbb{R}^2)$$

is given by

$$(3.3) \quad (\mathcal{F}_{\mathcal{H}}^{-1} h)(x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr}(h(\xi) e_{\xi}(x)), \quad h \in \mathcal{S}(\mathbb{N}^2),$$

so that the Fourier inversion formula becomes

$$(3.4) \quad f(x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr} \left(\widehat{f}(\xi) e_\xi(x) \right) \quad \text{for all } f \in C_{\mathcal{H}}^\infty(\mathbb{R}^2).$$

The Plancherel's identity taking the form

$$(3.5) \quad \|f\|_{L^2(\mathbb{R}^2)} = \left(\sum_{\xi \in \mathbb{N}^2} \|\widehat{f}(\xi)\|_{\text{HS}}^2 \right)^{1/2} =: \|\widehat{f}\|_{\ell^2(\mathbb{N}^2)},$$

which we take as the definition of the norm on the Hilbert space $\ell^2(\mathbb{N}^2)$, and where $\|\widehat{f}(\xi)\|_{\text{HS}}^2 = \text{Tr}(\widehat{f}(\xi)\widehat{f}(\xi))$ is the Hilbert–Schmidt norm of the matrix $\widehat{f}(\xi)$.

One can readily check that test functions and distributions on \mathbb{R}^2 can be characterised in terms of their Fourier coefficients. Thus, we have

$$f \in C_{\mathcal{H}}^\infty(\mathbb{R}^2) \iff \forall N \exists C_N \text{ such that } \|\widehat{f}(\xi)\|_{\text{HS}} \leq C_N \langle \xi \rangle^{-N} \text{ for all } \xi \in \mathbb{N}^2.$$

Also, for distributions, we have

$$u \in \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2) \iff \exists M \exists C \text{ such that } \|\widehat{u}(\xi)\|_{\text{HS}} \leq C \langle \xi \rangle^M \text{ for all } \xi \in \mathbb{N}^2.$$

In general, given a linear continuous operator $L : C_{\mathcal{H}}^\infty(\mathbb{R}^2) \rightarrow C_{\mathcal{H}}^\infty(\mathbb{R}^2)$ (or even $L : C_{\mathcal{H}}^\infty(\mathbb{R}^2) \rightarrow \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2)$), we can define its matrix symbol by $\sigma_L(x, \xi) := e_\xi(x)^{-1}(Le_\xi)(x) \in \mathbb{C}^{2 \times 2}$, where Le_ξ means that we apply L to the matrix components of $e_\xi(x)$, provided that $e_\xi(x)$ is invertible in a suitable sense. In this case we may prove that

$$(3.6) \quad Lf(x) = \sum_{\xi \in \mathbb{N}^2} \text{Tr} \left(e_\xi(x) \sigma_L(x, \xi) \widehat{f}(\xi) \right).$$

The correspondence between operators and symbols is one-to-one. The quantization (3.6) has been extensively studied in [RT10, RT13] in the setting of compact Lie groups, and in [RT16] in the setting of (non-self-adjoint) boundary value problems, to which we may refer for its properties and for the corresponding symbolic calculus. However, the situation with the Landau Hamiltonian is essentially simpler since this operator can be treated as an ‘invariant’ operator in the corresponding global calculus.

The operator \mathcal{H} is formally self-adjoint, therefore its symbol $\sigma_{\mathcal{H}}(\xi)$ is independent of x and can be diagonalised by a choice of the basis in the in \mathbb{R}^2 . Moreover, the operator \mathcal{H} is positive. Therefore, without loss of generality, we can always write

$$(3.7) \quad \sigma_{\mathcal{H}}(\xi) = \begin{pmatrix} \nu_1^2(\xi) & 0 \\ 0 & \nu_2^2(\xi) \end{pmatrix},$$

for some $\nu_j(\xi) \geq 0$. Indeed, we have $\nu_j^2(\xi) = B(1 + 2\xi_2)$ for $j = 1, 2$.

Consequently, we can also define Sobolev spaces $H_{\mathcal{H}}^s(G)$ associated to \mathcal{H} . Thus, for any $s \in \mathbb{R}$, we set

$$(3.8) \quad H_{\mathcal{H}}^s(G) := \{f \in \mathcal{D}'_{\mathcal{H}}(\mathbb{R}^2) : \mathcal{H}^{s/2} f \in L^2(\mathbb{R}^2)\},$$

with the norm $\|f\|_{H_{\mathcal{H}}^s} := \|\mathcal{H}^{s/2}f\|_{L^2}$. Using Plancherel's identity (3.5), we can write

$$(3.9) \quad \|f\|_{H_{\mathcal{H}}^s} = \|\mathcal{H}^{s/2}f\|_{L^2} = \left(\sum_{\xi \in \mathbb{N}^2} \|\sigma_{\mathcal{H}}(\xi)^{s/2} \widehat{f}(\xi)\|_{\text{HS}}^2 \right)^{1/2} \\ = \left(\sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^s \sum_{j=1}^2 |\widehat{f}(\xi)_{jj}|^2 \right)^{1/2} = \left(\sum_{\xi \in \mathbb{N}^2} (B + 2B\xi_2)^s \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} f(x) \overline{e_{\xi}^j(x)} dx \right|^2 \right)^{1/2},$$

justifying the expression (2.2).

4. PROOF OF THE THEOREM

The operator \mathcal{H} has the symbol (3.7), which we can write in matrix components as

$$\sigma_{\mathcal{H}}(\xi)_{mk} = B(1 + 2\xi_2)\delta_{mk}, \quad 1 \leq m, k \leq 2,$$

with δ_{mk} standing for the Kronecker's delta. Taking the \mathcal{H} -Fourier transform of (1.7), we obtain the collection of Cauchy problems for matrix-valued Fourier coefficients:

$$(4.1) \quad \partial_t^2 \widehat{u}(t, \xi) - a(t)\sigma_{\mathcal{H}}(\xi)\widehat{u}(t, \xi) = 0, \quad \xi \in \mathbb{N}^2.$$

Writing this in the matrix form, we see that this is equivalent to the system

$$\partial_t^2 \widehat{u}(t, \xi) + a(t) \begin{pmatrix} B(1 + 2\xi_2) & 0 \\ 0 & B(1 + 2\xi_2) \end{pmatrix} \widehat{u}(t, \xi) = 0,$$

where we put explicitly the diagonal symbol $\sigma_{\mathcal{H}}(\xi)$. Rewriting (4.1) in terms of matrix coefficients $\widehat{u}(t, \xi) = (\widehat{u}(t, \xi)_{mk})_{1 \leq m, k \leq 2}$, we get the equations

$$(4.2) \quad \partial_t^2 \widehat{u}(t, \xi)_{mk} + a(t)B(1 + 2\xi_2)\widehat{u}(t, \xi)_{mk} = 0, \quad \xi \in \mathbb{N}^2, \quad 1 \leq m, k \leq 2.$$

The main point of our further analysis is that we can make an individual treatment of the equations in (4.2). Thus, let us fix $\xi \in \mathbb{N}^2$ and m, k with $1 \leq m, k \leq 2$, and let us denote $\widehat{v}(t, \xi) := \widehat{u}(t, \xi)_{mk}$. We then study the Cauchy problem

$$(4.3) \quad \partial_t^2 \widehat{v}(t, \xi) + a(t)B(1 + 2\xi_2)\widehat{v}(t, \xi) = 0, \quad \widehat{v}(t, \xi) = \widehat{v}_0(\xi), \quad \partial_t \widehat{v}(t, \xi) = \widehat{v}_1(\xi),$$

with ξ, m being parameters, and want to derive estimates for $\widehat{v}(t, \xi)$. Combined with the characterisation (3.9) of Sobolevspaces this will yield the well-posedness results for the original Cauchy problem (1.7).

In the sequel, for fixed m , we set

$$(4.4) \quad |\xi|_{\nu}^2 := B(1 + 2\xi_2).$$

Hence, the equation in (4.3) can be written as

$$(4.5) \quad \partial_t^2 \widehat{v}(t, \xi) + a(t)|\xi|_{\nu}^2 \widehat{v}(t, \xi) = 0.$$

We now proceed with a standard reduction to a first order system of this equation and define the corresponding energy. The energy estimates will be given in terms of t and $|\xi|_{\nu}$ and we then go back to t, ξ and m by using (4.4).

We can now do the natural energy construction for (4.5). We use the transformation

$$\begin{aligned} V_1 &:= i|\xi|_\nu \widehat{v}, \\ V_2 &:= \partial_t \widehat{v}. \end{aligned}$$

It follows that the equation (4.5) can be written as the first order system

$$(4.6) \quad \partial_t V(t, \xi) = i|\xi|_\nu A(t) V(t, \xi),$$

where V is the column vector with entries V_1 and V_2 and

$$A(t) = \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}.$$

The initial conditions $\widehat{v}(0, \xi) = \widehat{v}_0(\xi)$, $\partial_t \widehat{v}(0, \xi) = \widehat{v}_1(\xi)$ are transformed into

$$V(0, \xi) = \begin{pmatrix} i|\xi|_\nu \widehat{v}_0(\xi) \\ \widehat{v}_1(\xi) \end{pmatrix}.$$

Note that the matrix A has eigenvalues $\pm \sqrt{a(t)}$ and symmetriser

$$S(t) = \begin{pmatrix} 2a(t) & 0 \\ 0 & 2 \end{pmatrix}.$$

By definition of the symmetriser we have that

$$SA - A^*S = 0.$$

It is immediate to prove that

$$(4.7) \quad 2 \min_{t \in [0, T]} (a(t), 1) |V|^2 \leq (SV, V) \leq 2 \max_{t \in [0, T]} (a(t), 1) |V|^2,$$

where (\cdot, \cdot) and $|\cdot|$ denote the inner product and the norm in \mathbb{C} , respectively.

Since $a(t) > 0$, $a \in C^1([0, T])$, it is clear that there exist constants $a_0 > 0$ and $a_1 > 0$ such that

$$a_0 = \min_{t \in [0, T]} a(t) \quad \text{and} \quad a_1 = \max_{t \in [0, T]} a(t).$$

Hence (4.7) implies,

$$(4.8) \quad c_0 |V|^2 = 2 \min(a_0, 1) |V|^2 \leq (SV, V) \leq 2 \max(a_1, 1) |V|^2 = c_1 |V|^2,$$

with $c_0, c_1 > 0$. We then define the energy

$$E(t, \xi) := (S(t)V(t, \xi), V(t, \xi)).$$

We get, from (4.8), that

$$\begin{aligned}
\partial_t E(t, \xi) &= (\partial_t S(t)V(t, \xi), V(t, \xi)) + (S(t)\partial_t V(t, \xi), V(t, \xi)) \\
&\quad + (S(t)V(t, \xi), \partial_t V(t, \xi)) \\
&= (\partial_t S(t)V(t, \xi), V(t, \xi)) + i|\xi|_\nu (S(t)A(t)V(t, \xi), V(t, \xi)) \\
&\quad - i|\xi|_\nu (S(t)V(t, \xi), A(t)V(t, \xi)) \\
&= (\partial_t S(t)V(t, \xi), V(t, \xi)) + i|\xi|_\nu ((SA - A^*S)(t)V(t, \xi), V(t, \xi)) \\
&= (\partial_t S(t)V(t, \xi), V(t, \xi)) \\
&\leq \|\partial_t S\| |V(t, \xi)|^2 \\
&\leq c'E(t, \xi)
\end{aligned}$$

i.e. we obtain

$$(4.9) \quad \partial_t E(t, \xi) \leq c'E(t, \xi),$$

for some constant $c' > 0$. By Gronwall's lemma applied to inequality (4.9) we conclude that for all $T > 0$ there exists $c > 0$ such that

$$E(t, \xi) \leq cE(0, \xi).$$

Hence, inequalities (4.8) yield

$$c_0 |V(t, \xi)|^2 \leq E(t, \xi) \leq cE(0, \xi) \leq cc_1 |V(0, \xi)|^2,$$

for constants independent of $t \in [0, T]$ and ξ . This allows us to write the following statement: there exists a constant $C_1 > 0$ such that

$$(4.10) \quad |V(t, \xi)| \leq C_1 |V(0, \xi)|,$$

for all $t \in [0, T]$ and ξ . Hence

$$|\xi|_\nu^2 |\widehat{v}(t, \xi)|^2 + |\partial_t \widehat{v}(t, \xi)|^2 \leq C'_1 (|\xi|_\nu^2 |\widehat{v}_0(\xi)|^2 + |\widehat{v}_1(\xi)|^2).$$

Recalling the notation $\widehat{v}(t, \xi) = \widehat{u}(t, \xi)_{mk}$ and $|\xi|_\nu = B(1 + 2\xi_2)$, this means

$$(4.11) \quad B(1 + 2\xi_2) |\widehat{u}(t, \xi)_{mk}|^2 + |\partial_t \widehat{u}(t, \xi)_{mk}|^2 \leq C'_1 (B(1 + 2\xi_2) |\widehat{u}_0(\xi)_{mk}|^2 + |\widehat{u}_1(\xi)_{mk}|^2)$$

for all $t \in [0, T]$, $\xi \in \mathbb{N}^2$ and $1 \leq m, k \leq 2$, with the constant C'_1 independent of ξ , m, k . Now we recall that by Plancherel's equality, we have

$$\|\partial_t u(t, \cdot)\|_{L^2}^2 = \sum_{\xi \in \mathbb{N}^2} \|\partial_t \widehat{u}(t, \xi)\|_{\text{HS}}^2 = \sum_{\xi \in \mathbb{N}^2} \sum_{m,k=1}^2 |\partial_t \widehat{u}(t, \xi)_{mk}|^2$$

and

$$\|\mathcal{H}u(t, \cdot)\|_{L^2}^2 = \sum_{\xi \in \mathbb{N}^2} \|\sigma_{\mathcal{H}}(\xi) \widehat{u}(t, \xi)\|_{\text{HS}}^2 = \sum_{\xi \in \mathbb{N}^2} \sum_{m,k=1}^2 B(1 + 2\xi_2) |\widehat{u}(t, \xi)_{mk}|^2.$$

Hence, the estimate (4.11) implies that

$$(4.12) \quad \|\mathcal{H}u(t, \cdot)\|_{L^2}^2 + \|\partial_t u(t, \cdot)\|_{L^2}^2 \leq C(\|\mathcal{H}u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2),$$

where the constant $C > 0$ does not depend on $t \in [0, T]$. More generally, multiplying (4.11) by powers of $(B + 2B\xi_2)$, for any s , we get

$$(4.13) \quad (B + 2B\xi_2)^{1+s} |\widehat{u}(t, \xi)_{mk}|^2 + (B + 2B\xi_2)^s |\partial_t \widehat{u}(t, \xi)_{mk}|^2 \\ \leq C'_1 ((B + 2B\xi_2)^{1+s} |\widehat{u}_0(\xi)_{mk}|^2 + (B + 2B\xi_2)^s |\widehat{u}_1(\xi)_{mk}|^2).$$

Taking the sum over ξ , m and k as above, this yields the estimate (2.1).

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